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Research and Pedagogies for Early Math

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Abstract: The increasing interest in early childhood mathematics education for decades has increased the need for empirically supported pedagogical strategies. However, there is little agreement on how

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teaching practices to student outcomes, backgrounds and characteristics of students, different ways students engage and process what teachers present, and so on. Although these have all contributed to knowledge, they generally have not found practically meaningful links between specific teacher actions and student learning outcomes, e.g., [41–43] and analyses from the complex array of variables across these types of studies similarly provide little guidance. What *can* guide teaching? Understanding how to provide adequate high-quality educative experiences to achieve mathematical learning goals (which may stem from standards, or from recognition of a child's interests and needs). Thus, teachers can consistently focus on creating research-based, empirically validated Sustained Learning Opportunities (SLO) [44]. Educators develop and plan fecund instructional tasks and patterns of interactions, realized by teachers and children collaboratively so they are meaningful to all. These are based on teachers' understanding of students' levels of thinking and their development across SLOs, progressing toward the educational goal [5,39]. Students' intellectual work occurs within the triarchic interaction of teachers, students, and mathematics content and activities.

Finally, as useful as these findings are, they are general teaching strategies. High-quality teaching also depends on knowledge of content, how children think and learn about that content, and how specifically to teach that content for each important topic in early math [39,45–54]. This applies to intentional teaching, and perhaps more so to child-initiated contexts such as play, so as to fully understand how to support children's creative math thinking and learning [5,39,55]. We ground our interpretations within our theory of Hierarchical Interactionalism, a synthesis of empiricist, nativist, and especially constructivist theories, that emphasizes these three knowledge domains [40,56].

3. Children's Learning with Different Approaches to Teaching

As one of the most complex human enterprises, teaching is difficult to define and study. Here, we define the teaching of math as intentional interactions among children and teachers around mathematics content using deliberately arranged environments, contexts, and tasks, all designed to promote children's learning of increasingly powerful and sophisticated math competencies and positive dispositions. Those goals—competencies and dispositions—lead to our first issue.

3.1. General Teaching Approaches for Different Goals

When not recognized, differences in these goals can lead us to believe that research is contradictory when it is not because different pedagogical approaches can be effective for different goals [41]. For example, when learning skills, or targeting instrumental understanding (rules without reasons) [57], is the primary goal, certain teaching strategies, such as whole group organization, clear directions and explanations with modeling, fast pacing, emphasis on mastery, and careful review are effective [58–62]. In contracal contracal contracal contr

learning goal, ensuring children have a degree of choice and agency, and using their understanding of children's thinking and interests to choose strategies, such as open-ended questions, hints, prompts, and modeling [91,100].

Finally, a playful but intentional teaching approach is more effective in promoting math learning than laissez-fair approaches or teaching based only on "teachable moments" [55,89,101–103], including in free play contexts, such as the block center [88,104]. This is especially true for children with disabilities [105]. Later sections address intentional teaching.

Unsurprisingly, these issues and suggestions mirror similar findings in the debates on discovery learning, in which unguided discovery is more effective than guided discovery teaching [60,106–108] and better at developing concepts that direct instruction alone [109].

However, even direct instruction can play an important role in a multidimensional pedagogical toolkit, especially at appropriate junctures with discovery- or inquiry-based learning contexts [109,110]. As a simple example, direct instruction is necessary and efficient for Piaget's social-arbitrary knowledge, such as spelling "four", writing "4" or other mathematical symbols, conventions, or simple procedures. Physical knowledge is learned by activity on objects. In contrast, logical-mathematics knowledge is learned from *thinking about* one's actions on the objects [111]. Intentional, playful experiences and guided discovery approaches develop deep understanding and transfer needed for relational understanding in all math topics [39,112]. Strategies from the pedagogical toolkit are best deployed depending on the content, context, and children. For example, children who explore math ideas playfully before intentional instruction use a greater variety of strategies and attend to the features of problems more than those instructed first [113].

In summary, those teaching for relational understanding view children as active learners who initiate explorations of and interactions with the surrounding world and both adults and peers [26,33,114–121]. They avoid a preponderance of passive "reception" of knowledge, understanding that children construct knowledge from a wide variety of experiences [122], including direct instruction when it contributes to their learning. Such experiences support learning and development and minimize wasted time in passive experiences such as waiting [123]. Teachers support learning by using an equity lens to watch and listen to children and the way they express their ideas [76]. By observing, interacting, and being reflective, they base interactions and activities on children's thinking and learning [114,120]. In these ways, they promote joyful, engaged learning for all children [124] from birth (we recognize that space limitations did not allow addressing infants and toddlers) [33], we recognize that space limitations did not allow addressing infants and toddlers.

3.3. Intentional Teaching and the Central Role of Children's Thinking and Learning

A critical feature of teaching approaches that develop relational thinking is that they actions and strategies to solve the problem are represented. There is reflection on whether the problem is solved, or partially solved, which leads to new understandings (mental actions and objects, organized into strategies and structures) and actions [54]. Specific learning trajectories are the main bridge that connect the "grand theory" of Hierarchic Interactionalism [40] to particular theories and educational practice.

All three components of a LT can be misunderstood. Table 1 addresses misunderstandings and myths to make the theory and its application clear.

Table 1. Myths and Understandings of Learning Trajectories.

Component	Misunderstanding/Myths	True Learning Trajectories
Goal	Narrow behavioral objective	"Big ideas and proficiencies, central and coherent, consistent with children's thinking, and generative of future learning. Math practices and investigations [33] Positive dispositions
Developmental Progression (DP)	Rigid sequence of skills in "small steps"	Broad levels of learning; patterns of thinking including concepts and structures [31,54], skills, practices, etc.
Instructional Activities	Rote-skill based or Generic	Connected to each level of the DP–concepts, skills, and problem solving. Designed to promote thinking at that level–the actions-on-objects (often right in the activity—unitizing, composing, etc.)
Learning Trajectories	Break down skills into sequences, all followed in lock step	*Building up* children from and through their natural ways of thinking (asset-based) [32].

The LT approach has been research validated in multiple studies for a wide variety of math topics [32,92,127–135]. In most, teachers used all the strategies in the previously described multidimensional pedagogical toolkit. Further, they combined brief, active, whole-group sessions, individual work (sometimes using educational technology), incidental learning throughout the day, and small-group sessions. The last was especially important due to the personal involvement and close interactions, supporting their understanding and *use* of children's thinking to differentiate instruction. Such *formative assessment* is one of the most strongly empirically supported teaching approaches [67,136,137]. Formative assessment is the ongoing understanding of children's thinking and learning to inform and adapt instruction for groups and individuals. However, formative assessment is not useful if teaching is not adapted based on that understanding [67,138]. Effective teachers ask and answer the following questions: what do children need to learn, where are children now, and how do I help them progress? [137]. Importantly, these questions align with the three components of LTs: goal, developmental progression, and linked teaching activities and strategies. This may be why LTs support and contribute to teachers' professional development and teaching prowess [139–141] and children's learning [92,127,133,142,143].

Considering the validating studies cited, it is important to note that many have involved a specific curriculum, so that the LTs may have been confounded by other differences between the compared groups. Therefore, studies that rigorously compared LT-based instruction to the same instruction without a critical aspect of LTs address their specific

In summary, teachers who know how to use the three components of a learning trajectory are more effective in supporting children's learning [150]. Without such knowledge, teachers of young children might offer tasks that are either too easy or too hard for children, and this mismatch may limit children's learning [39,151]. Playful, meaningful, content-rich education based on learning trajectories benefits all children. Indeed, it is especially important for children with disabilities (CWD) [152]. CWD might operate at levels different from their peers and quite different levels in one topic (say, counting) than others (such as geometry). Learning trajectories offer different ways to introduce math topics, such as arithmetic (e.g., counting, subitizing, or partitioning), so children can build on their individual strengths. Learning trajectories' levels are clusters of ideas and processes, not just skills. So, children can both learn and show competence using a variety of modalities and representations. Finally, learning trajectories can be aligned with formative assessment and the Individualized Education Program (IEP) or the Individualized Family Service Plan (IFSP) process. For all children with disabilities or math difficulties, tiered support is important and validated as effective [153,154].

The remainder of this section consists of brief reviews of specific teaching strategies for relational understanding. We start with additional research on formative assessment.

3.4. Formative Assessment

Formative assessment, the ongoing monitoring of student learning to inform instruction, was mentioned previously. Of the 10 instructional practices the National Mathematics Advisory Panel (NMAP) researched, only a few had an adequate number of rigorous studies supporting them. One of the most strongly supported was teachers' useteacher(start) ieitiesc

Challenging non-examples of shapes can be paired with an example, such as a triangle next to a visually-similar quadrilateral with one short side [39].

A study of examples in arithmetic found that second-grade children notice structure by analyzing worked examples, and they try to make sense of them based on prior knowledge [191]. Therefore, the first worked example *contrast* is important as it confirms or challenges their prior understanding. For example, using contrasting cases, such as having students compare the problem $5 - 3 = 2$ with $3 - 5 = -2$, can help them notice important features. Another study confirms the benefits of teachers asking children to compare and contrast ways of reasoning used on problems of different problem types to evoke different strategies [50]. Children then see and understand features of each problem that made one way of reasoning easier for solving one problem type than another. Such comparisons develop better problem solving and flexible mathematical thinking [50].

3.9. "Concrete" Manipulatives for "Abstract" Ideas

Teachers often move from "concrete" (e.g., using manipulatives) to "abstract" experiences for children. Although generally research supports this sequence, there are some critical nuances [125,192]. As an example of a study validating the approach, second graders randomly assigned to be taught with manipulatives achieved and retained significantly more on a place value comprehension test than students assigned to be taught by conventional methods using algorithmic procedures and drill and practice [193]. In addition, a case study of third graders with disabilities showed a relationship between the sequence and a place value assessment, including generalization to new tasks [194]. For example, just providing connecting cubes increased the math scores of second graders [195].

However, manipulatives do not guarantee success. Students taught multiplication emphasizing understanding performed well whether they used manipulatives or symbols [196,197]. Further, the students randomly assigned to be taught with symbols scored higher on an immediate transfer test involving different factors [198]. Manipulatives do not "carry" mathematical ideas. *They may help in teaching concretely at first, but only if such concrete teaching emphasizes quantitative or spatial ideas.*

Why might concrete manipulatives help? The answer has an interesting twist. Many would say that because they are physical objects that students can grasp with their hands, this sensory characteristic makes manipulatives "real", connected with one's intuitively meaningful personal self, and therefore helpful. However, concepts cannot be "read off" manipulatives. Expert teacher John Holt said that he and his fellow teacher "were excited about the [Cuisenaire] rods because we could see strong connections between the world of rods and the world of numbers. We therefore assumed that children, looking at the rods and doing things with them, could see how the world of numbers and numerical operations worked. The trouble with this theory is that [my colleague] and I already knew how the numbers worked. We could say, 'Oh, the rods behaved just the way numbers do'. But if we hadn't known how numbers behaved, would looking at the rods enable us to find out? Maybe so, maybe not" (Holt 1982, pp. 138–139). That is, the physical objects may be manipulated without the concepts being illuminated. Concrete materials may help students build meaning, but the students must reflect on their actions with manipulatives. Said in another way, "understanding does not travel through the fingertips and up the arm". [199] (p. 47). They need teachers to reflect on their students' representations for mathematical ideas and help them develop increasingly sophisticated and mathematical representations.

Children have Sensory-Concrete knowledge when they need to use sensory material to make sense of an idea [197,200]. For example, very young children need to count objects they can see to count meaningfully [39]. Later, teachers can help them develop Integrated-Concrete knowledge that connects concrete experiences to more abstract math concepts. There is a shift in what the adjective "concrete" describes. Sensory-Concrete refers to knowledge that demands the support of concrete objects and students' knowledge of manipulating these objects. Integrated-Concrete refers to knowledge that is concrete at a higher level because it is connected to other knowledge, both physical knowledge that has been abstracted and thus distanced from concrete objects and abstract knowledge of a variety of types.

Multiple studies have shown the benefit of supporting children in progressing from Sensory-Concrete to Integrated-Concrete cognition [192,200,201]. Usually, teachers first develop children's Sensory-Concrete implicit levels of thinking, at which perceptual supports are necessary and fundamental, and reasoning may be restricted to limited cases. Then they learn explicit

3.10. Practice

Research shows that teaching for relational understanding also develops skills. That does not mean young children do not need practice [214]. Fortunately, research offers clear guidelines. Rather than substantial time spent on drill, repeated experiences with many contexts and different types of activities support generalization and transfer [40,215]. Moreover, distributed, spaced practice is better than massed (all in one session, repetition of the same item repeatedly) practice [216,217]. Unfortunately, such practice is hotly debated.

Contrary to those who believe practice has no role and the so-called "science of math" movement that promotes memorization through drill without caveats, practice should be used at the correct developmental juncture and to the appropriate degree [218]. Because competencies in subitizing, counting, and arithmetic combinations support math thinking and learning throughout life, short, frequent practice sessions of facts and skills *whose conceptual foundations have been well learned and understood* are recommended. Finally, a classic conceptualization describes three levels of practice: the level of drill, application, or problem solving [219]. Practice at the problem-solving level teaches all the competencies of relational understanding. Meaningful practice develops more abilities and superior skills [181].

3.11. Affect, Motivation, and Engagement

Recall that productive disposition was one of the goals of relational understanding: a consistent view of math as sensible, useful, and worthwhile and of oneself as capable and engaged [63]. Contrary to this goal, one US cultural belief is that math achievement depends mostly on native aptitude or ability. In contrast, people from other countries, such as Japan, believe that achievement comes from effort [220]. Even more disturbing, research shows that the US belief hurts children and is not *valid.* Students who believe—or are helped to understand—that they can learn if they try to work on tasks longer and achieve better throughout their school careers than students who believe that one either "has it" (or "gets it") or does not [221]. This view often leads to failure, anxiety, and "learned helplessness" [221,222]. Similarly, students who have mastery-oriented goals (i.e., students who try to learn and see the point of school to develop knowledge and skills) achieve more than students whose goals are directed toward high grades or outperforming others [67,223].

Children's math anxiety predicts future math achievement over and above cognitive math ability, especially tackling challenging problems [224]. This adverse effect may be through children's visuospatial system [225]. Surprisingly, children with high achievement and high working memory may avoid using more advanced solution strategies *due* to math anxiety [226]. Unsurprisingly, most of these mirror the pedagogical strategies discussed previously, but the point here is that these have *also* been identified as improving children's attitudes and beliefs about math.

Fortunately, most young children have positive feelings about math and are motivated to explore numbers and shapes [223]. However, after only a couple of years in typical schools, they begin to believe that "only some people have the ability to do math". Children who experience math as a sense-making activity will build positive feelings about math throughout their school careers.

Teachers can help by providing meaningful tasks that make sense to students and connect with their everyday interests and lives. The right degree of challenge and novelty can promote interest, and promoting and discussing skill improvement can promote a mastery orientation. For example, researchers have estimated that students should be successful about 70% of the time to maximize motivation [223].

A common core of characteristics of learning environments enhances students' attitudes and beliefs about mathematics [227–233].

Use problems that have meaning for children (both practical and mathematical). (Note that even instruction that increases, for example, memorization via drill in the short run, may damage children's motivation.)

Expect that children will invent, explain, and critique their solution strategies within a social context.
Provide opportunities for creative invention and practice and promote inquiry [234].
Use manipulatives [195,235].
Use technology [235–237].
Encourage and support children progressing toward increasingly sophisticated and

Teachers held high standards for children's academic work but provided tools and strategies for students who needed extra support to reach proficiency.

Building upon Ladson-Billings and others' work, Gay [257] defined Culturally Responsive Teaching (CRT) with eight descriptors: validating, comprehensive and inclusive, cation of music could be used in math, e.g., patterning, [92,262] and related fields such as stance towards the children in [their] classroom[s]" (p. 224) including examining their students to move from passive recipients of knowledge to active learners and mentors who are developing confidence and self-efficacy.

Early childhood educators can also invite parents and caregivers to the classroom to serve as experts and capitalize on funds of knowledge in the community [277]. Parents and caregivers can teach skills, crafts, lead culturally specific activities, read books, and support math instruction.

4.6. Learning about How Math Is Taught in Other Cultures and Countries

Teachers who do not engage in culturally responsive mathematics education may hold the belief that mathematics is culturally neutral [278]. Teacher education and professional development sessions should work to deconstruct teachers' views that may include "beliefs about mathematics as a culturally-neutral subject, as universal truth, as a non-reasoning system, and, as an exclusively European and Western discipline" [278] (p. 51). By learning about the differences in how mathematics is taught and learned in countries and cultures around the world, early childhood educators can understand and appreciate differences in students' thinking and provide students with multiple strategies to approach their learning of mathematics.

An alternative teaching strategy that was used by early childhood educators in India involved having the children use their fingers to count in ways different from other cultures [279]. Children begin with the fingers of one hand, then the fingers of both hands, and then extended to using the joints and finger lines of both hands for a total count of 40. If students or their families have recently immigrated, early childhood educators can further experience" [283] (p. 25). For example, mis-educative experiences resulting from inappropriate direct teaching may decrease sensitivity to the wide range of applications of math ideas or develop automatic skills but narrow the range of other experiences with the idea underlying the skill. Conversely, child-centered education that rejects the structures or sequencing of subject matter content may be so disconnected as to limit later integrative experiences. As Dewey said, "Just because traditional education was a matter of routine in which the plans and programs were handed down from the past, it does not follow that progressive education is a matter of planless improvisation" (p. 28).

Regardless of instructional approach or strategy, educators must remember that young children's ideas can be uniquely different from those of adults [31,39,284,285]. Early childhood teachers must be careful not to assume that children "see" situations, problems, or solutions as adults do. Based on their interpretations of children's thinking, teachers conjecture what the child might be able to learn or abstract from his or her experiences. Similarly, when interacting with the child, they also consider their own actions from the child's point of view. This makes early childhood teaching both demanding and rewarding. Such sensitivity, however, is necessary to fully benefit from this chapter's pedagogical suggestions, especially the core contention of the central role of children's thinking and learning, as well as the use of formative assessment, and a variety of teaching strategies at each particular phase of learning. Knowledge of developmental paths in learning trajectories can enhance teachers' understanding of children's thinking, helping teachers assess children's level of understanding and offer instructional activities at the next level and thus offer meaningful and joyful opportunities to engage in learning.

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References

1. Parks, A.N. Centering children in mathematics education classroom research. *Am. Educ. Res. J.* **2019**, *57*, 1443–1484. [CrossRef]
2. Gerofsky, P.R. Why Asian preschool children mathematically outperform preschool children from other countries. *West. Undergrad. Psychol. J.* **2015**, *3*.
3. OECD. *Strong Performers and Successful Reformers in Education—Lessons from PISA 2012 for the United States*; OECD Publishing: Paris, France, 2014. [CrossRef]
4. Bachman, H.J.; Votruba-Drzal, E.; El Nokali, N.E.; Castle Heatly, M. Opportunities for learning math in elementary school: Implications for SES disparities in procedural and conceptual math skills. *Am. Educ. Res. J.* **2015**, *52*, 894–923. [CrossRef]
5. Gripton, C.; Williams, H.J. The principles for appropriate pedagogy in early mathematics: Exploration, apprenticeship and sense-making: Part 2. *Math. Teach.* **2023**, *286*, 5–7.
6. McCoy, D.C.; Salhi, C.; Yoshikawa, H.; Black, M.; Britto, P.; Fink, G. Home- and center-based learning opportunities for preschoolers in low- and middle-income countries. *Child. Youth Serv. Rev.* **2018**, *88*, 44–56. [CrossRef]
7. Strunk, K.O.; Bradshaw, C.P.; Wijekumar, K.; Therriault, S.B. Disproportionate impact of COVID-19 on student learning and contributions of education sciences to pandemic recovery efforts. In Proceedings of the IES Principal Investigators Meeting: Building on 20 Years of ies Research to Accelerate the Education Sciences, Washington, DC, USA, 16 May 2023.
8. Galindo, C.; Sonnenschein, S. Decreasing the SES math achievement gap: Initial math proficiency and home learning environments. *Contemp. Educ. Psychol.* **2015**, *43*, 25–38. [CrossRef]

9. Sonnenschein, S.; Baker, L.; Moyer, A.; LeFevre, S. Parental beliefs about children's reading and math development and relations with subsequent achievement. In Proceedings of the Biennial Meeting of the Society for Research in Child Development, Atlanta, GA, USA, 7–10 April 2005.
10. McCoy, D.C.; Yoshikawa, H.; Ziol-Guest, K.M.; Duncan, G.J.; Schindler, H.S.; Magnuson, K.; Yang, R.; Koepp, A.; Shonkoff, J.P. Impacts of early childhood education on medium- and long-term educational outcomes. *Educ. Res.* **2017**, *46*, 474–487. [CrossRef] [PubMed]
11. Watts, T.W.; Duncan, G.J.; Siegler, R.S.; Davis-Kean, P.E. What's past is prologue: Relations between early mathematics knowledge and high school achievement. *Educ. Res.* **2014**, *43*, 352–360. [CrossRef]
12. Ritchie, S.J.; Bates, T.C. Enduring links from childhood mathematics and reading achievement to adult socioeconomic status. *Psychol. Sci.* **2013**, *24*, 1301–1308. [CrossRef]
13. Duncan, G.J.; Magnuson, K. The nature and impact of early achievement skills, attention skills, and behavior problems. In *Whither Opportunity? Rising Inequality and the Uncertain Life Chances of Low-Income Children*; Duncan, G.J., Murnane, R., Eds.; Sage: New York, NY, USA, 2011; pp. 47–70.
14. Sarama, J.; Lange, A.; Clements, D.H.; Wolfe, C.B. The impacts of an early mathematics curriculum on emerging literacy and language.

38. Clements, D.H.; Sarama, J. Early childhood mathematics learning. In *Second Handbook of Research on Mathematics Teaching and Learning*; Lester, F.K., Jr., Ed.; Information Age Publishing: New York, NY, USA, 2007; Volume 1, pp. 461–555.
39. Clements, D.H.; Sarama, J. *Learning and Teaching Early Math: The Learning Trajectories Approach*; Routledge: New York, NY, USA, 2009.
40. Sarama, J.; Clements, D.H. *Early Childhood Mathematics Education Research: Learning Trajectories for Young Children*; Routledge: New York, NY, 2009.
41. Hiebert, J.C.; Grouws, D.A. The effects of classroom mathematics teaching on students' learning. In *Second Handbook of Research on Mathematics Teaching and Learning*; Lester, F.K., Jr., Ed.; Information Age Publishing: New York, NY, USA, 2007; Volume 1, pp. 371–404.
42. Kane, T.J.; Taylor, E.S.; Tyler, J.H.; Wooten, A.L. Identifying effective classroom practices using student achievement data. *J. Hum. Resour.* **2011**, *46*, 587–613.
43. Pemu, N. Connecting teacher's activity and students' achievement toward numeracy competency in learning mathematics. *Pedagog.—Theory Prax.* **2023**, *8*, 196–211.
44. Hiebert, J.; Stigler, J.W. Creating practical theories of teaching. In *Theorizing Teaching: Current Status and Open Issues*; Praetorius, A.-K., Charalambous, C.Y., Eds.; Springer International Publishing: Cham, Switzelrand, 2023; pp. 23–56. [CrossRef]
45. Bojorque, G.; Torbeyns, J.; Hannula-Sormunen, M.; Van Nijlen, D.; Verschaffel, L. Ecuadorian kindergartners' spontaneous focusing on numerosity development: Contribution of numerical abilities and quality of mathematics education. In *Contemporary Research and Perspectives on Early Childhood Mathematics Education*; Elia, I., Mulligan, J., Anderson, A., Baccaglini-Frank, A., Benz, C., Eds.; Springer International Publishing: Cham, Switzelrand, 2018; pp. 69–86. [CrossRef]
46. Carpenter, T.P.; Franke, M.L.; Levi, L. *Thinking Mathematically: Integrating Arithmetic and Algebra in Elementary School*; Heinemann: Portsmouth, NH, USA, 2003.
47. Cheeseman, J.; Benz, C.; Pullen, Y. Number sense: The impact of a measurement-focused program on young children's number learning. In *Contemporary Research and Perspectives on Early Childhood Mathematics Education*; Elia, I., Mulligan, J., Anderson, A., Baccaglini-Frank, A., Benz, C., Eds.; Springer International Publishing: Cham, Switzelrand, 2018; pp. 101–127. [CrossRef]
48. Confrey, J.; Maloney, A.P.; Nguyen, K.H.; Rupp, A.A. Equipartitioning: A foundation for rational number reasoning. Elucidation of a learning trajectory. In *Learning over Time: Learning Trajectories in Mathematics Education*; Maloney, A.P., Confrey, J., Nguyen, K.H., Eds.; Information Age Publishing: New York, NY, USA, 2014; pp. 61–96.
49. Dyson, N.; Jordan, N.C.; Beliakoff, A.; Hassinger-Das, B. A kindergarten number-sense intervention with contrasting practice conditions for low-achieving children. *J. Res. Math. Educ.* **2015**, *46*, 331–370. [CrossRef] [PubMed]
50. Lamb, L.; Bishop, J.; Whitacre, I.; Philipp, R. Flexibility across and flexibility within: The domain of integer addition and subtraction. *J. Math. Behav.* **2023**, *70*, 101031. [CrossRef]
51. Lüken, M.M. Repeating pattern competencies in three- to five-year old kindergartners: A closer look at strategies. In *Contemporary Research and Perspectives on Early Childhood Mathematics Education*; Elia, I., Mulligan, J., Anderson, A., Baccaglini-Frank, A., Benz, C., Eds.; Springer International Publishing: Cham, Switzerland, 2018; pp. 35–53. [CrossRef]
52. Novotná, J.; Kaur, B.; Gervasoni, A.; Askew, M.; Veldhuis, M.; Pearn, C.; Sun, X.H. How to teach and assess whole number arithmetic: Some international perspectives. In *Building the Foundation: Whole Numbers in the Primary Grades: The 23rd ICMI Study*;

62. Heasty, M.; McLaughlin, T.F.; Williams, R.L.; Keenan, B. The effects of using direct instruction mathematics formats to teach basic math skills to a third grade student with a learning disability. *Acad. Res. Int.* **2012**, *2*, 382–387.
63. National Research Council. *Adding It Up: Helping Children Learn Mathematics*; National Academy Press: Washington, DC, USA, 2001. [CrossRef]
64. Doabler, C.T.; Nelson, N.J.; Kennedy, P.C.; Stoolmiller, M.; Fien, H.; Clarke, B.; Gearin, B.; Smolkowski, K.; Baker, S.K. Investigating the longitudinal effects of a core mathematics program on evidence-based teaching practices in mathematics. *Learn. Disabil. Q.* **2018**, *41*, 144–158. [CrossRef]
65. Fuson, K.C.; Briars, D.J. Using a base-ten blocks learning/teaching approach for first- and second-grade place-value and multidigit addition and subtraction. *J. Res. Math. Educ.* **1990**, *21*, 180–206. [CrossRef]
66. Gilmore, C.; Keeble, S.; Richardson, S.; Cragg, L. The interaction of procedural skill, conceptual understanding and working memory in early mathematics achievement. *J. Numer. Cogn.* **2017**, *3*, 400–416. [CrossRef]
67. National Mathematics Advisory Panel. *Foundations for Success: The Final Report of the National Mathematics Advisory Panel*; U.S. Department of Education, Office of Planning, Evaluation and Policy Development: Washington DC, USA, 2008.
68. Özcan, Z.Ç.; Doğan, H. A longitudinal study of early math skills, reading comprehension and mathematical problem solving. *Pegem Eğitim Ve Öğretim Derg.* **2017**, *8*, 1–18. [CrossRef]
69. Blöte, A.W.; van der Burg, E.; Klein, A.S. Students' flexibility in solving two-digit addition and subtraction problems: Instruction effects. *J. Educ. Psychol.* **2001**, *93*, 627–638. [CrossRef]
70. Knapp, M.S.; Shields, P.M.; Turnbull, B.J. *Academic Challenge for the Children of Poverty*; U.S. Department of Education: Washington, DC, USA, 1992.
71. Zigler, E.; Gilliam, W.S.; Barnett, W.S. (Eds.) *The Pre-K Debates: Current Controversies and Issues*; Brookes: Baltimore, MD, USA, 2011.
72. Clements, D.H.; Sarama, J. Play, mathematics, and false dichotomies. In *Preschool Matters. . .Today!* Barnett, W.S., Ed.; National Institute for Early Education Research (NIEER) at Rutgers University: New Brunswick NJ, USA, 2014; Volume 2014.
73. Kagan, S.L.; Reid, J.L. Reaching for consensus about preschool curricula. *Kappan* **2022**, *104*, 50–55.
74. Hartman, J.R.; Hart, S.; Nelson, E.A.; Kirschner, P.A. Designing mathematics standards in agreement with science. *Int. Electron. J. Math. Educ.* **2023**, *18*, em0739. [CrossRef]
75. Xu, C.; Di Lonardo Burr, S.; LeFevre, J.-A. The hierarchical relations among mathematical competencies: From fundamental numeracy to complex mathematical skills. *Can. J. Exp. Psychol.* **2023**, *in press*. [CrossRef] [PubMed]
76. Delpit, L. The silenced dialogue: Power and pedagogy in educating other people's children. *Harv. Educ. Rev.* **1988**, *58*, 280–298. [CrossRef]
77. Delpit, L. *"Multiplication Is for White People": Raising Expectations for Other People's Children*; The New Press: New York, NY, USA, 2012.
78. Mulligan, J.T.; Oslington, G.; English, L.D. Supporting early mathematical development through a 'pattern and structure' intervention program. *ZDM—Math. Educ.* **2020**, *52*, 663–676. [CrossRef]
79. Burger, K. Effective early childhood care and education: Successful approaches and didactic strategies for fostering child development. *Eur. Early Child. Educ. Res. J.* **2015**, *23*, 743–760. [CrossRef]
80. Merkley, R.; Ansari, D. *Foundations for Learning: Guided Play for Early Years Maths Education*; Chartered College of Teaching: London, UK, 2018.
81. Zosh, J.M.; Hirsh-Pasek, K.; Golinkoff, R.M. Playing to learn mathematics. In *Play-Based Learning*; Pyle, A., Ed.; Encyclopedia on Early Childhood Development: Montreal, QC, Canada, 2018; pp. 33–37.
82. Campbell, C.; Speldewinde, C.; Howitt, C.; MacDonald, A. STEM practice in the early years. *Creat. Educ.* **2018**, *9*, 11–25. [CrossRef]
83. Seo, K.-H.; Ginsburg, H.P. What is developmentally appropriate in early childhood mathematics education? In *Engaging Young Children in Mathematics: Standards for Early Childhood Mathematics Education*; Clements, D.H., Sarama, J., DiBiase, A.-M., Eds.; Erlbaum: Mahwah, NJ, USA, 2004; pp. 91–104.
84. Reikerås, E. Central skills in toddlers' and pre-schoolers' mathematical development, observed in play and everyday activities. *Nord. Stud. Math. Educ.* **2016**, *21*, 57–77.
85. Sim, Z.L.; Xu, F. Learning higher-order generalizations through free play: Evidence from 2- and 3-year-old children. *Dev. Psychol.* **2017**, *53*, 642–651. [CrossRef] [PubMed]
86. Helenius, O.; Johansson, M.L.; Lange, T.; Meaney, T.; Wernberg, A. Measuring temperature within the didaktic space of preschool. *Nord. Stud. Math. Educ.* **2016**, *21*, 155–176.
87.

91. Skene, K.; O'Farrelly, C.M.; Byrne, E.M.; Kirby, N.; Stevens, E.C.; Ramchandani, P.G. Can guidance during play enhance children's learning and development in educational contexts? A systematic review and meta-analysis. *Child Dev.* **2022**, *93*, 1162–1180. [CrossRef]
92. Clements, D.H.; Sarama, J. Effects of a preschool mathematics curriculum: Summative research on the Building Blocks project. *J. Res. Math. Educ.* **2007**, *38*, 136–163. [CrossRef]
93. Lewis Presser, A.; Clements, M.; Ginsburg, H.P.; Ertle, B. Big Math for Little Kids: The effectiveness of a preschool and kindergarten mathematics curriculum. *Early Educ. Dev.* **2015**, *26*, 399–426. [CrossRef]
94. Fidjeland, A.; Rege, M.; Solli, I.F.; Størksen, I. Reducing the gender gap in early learning: Evidence from a field experiment in Norwegian preschools. *Eur. Econ. Rev.* **2023**, *154*, 104413. [CrossRef]
95. Finders, J.K.; Duncan, R.J.; Purpura, D.J.; Elicker, J.; Schmitt, S.A. Testing theoretical explanations for heterogeneity in associations between a state quality rating and improvement system and prekindergarten children's academic performance. *Contemp. Educ. Psychol.* **2023**, *73*, 102174. [CrossRef]
96. Fuller, B.; Bein, E.; Bridges, M.; Kim, Y.; Rabe-Hesketh, S. Do academic preschools yield stronger benefits? Cognitive emphasis, dosage, and early learning. *J. Appl. Dev. Psychol.* **2017**, *52*, 1–11. [CrossRef]
97. Le, V.-N.; Schaack, D.; Neishi, K.; Hernandez, M.W.; Blank, R.K. Advanced content coverage at kindergarten: Are there trade-offs between academic achievement and social-emotional skills? *Am. Educ. Res. J.* **2019**, *56*, 1254–1280. [CrossRef]
98. Pound, L. Count on play: The importance of play in making sense of mathematics. In *Young Children's Play and Creativity: Multiple Voices*; Goodliff, G., Canning, N., Parry, J., Miller, L., Eds.; Routledge: Abingdon, UK; New York, NY, USA, 2017; pp. 220–228.
99. Van Oers, B.; Poland, M. Promoting abstract thinking in young children's play. In *Developmental Education for Young Children*; van Oers, B., Ed.; Springer: Dordrecht, The Netherlands, 2012; Volume 7, pp. 121–136.
100. Gervasoni, A. The impact and challenges of early mathematics intervention in an Australian context. In Proceedings of the 13th International Congress on Mathematical Education, 24–31 July 2016, Hamburg, Germany; Kaiser, G., Forgasz, H., Gravenm, M.,

117. Fröbel, F.W.A. *The Education of Man*; A. Lovell and Co.: New York, NY, USA, 1885.
118. Gelman, R. Constructivism and supporting environments. In *Implicit and Explicit Knowledge: An Educational Approach*; Tirosh, D., Ed.; Ablex: Norwood, NJ, USA, 1994; Volume 6, pp. 55–82.
119. National Research Council. *Eager to Learn: Educating our Preschoolers*; National Academy Press: Washington, DC, USA, 2001.
120. Samuelsson, I.P.; Sheridan, S.; Williams, P. Five preschool curricula—Comparative perspective. *Int. J. Early Child.* **2006**, *38*, 11–30. [CrossRef]
121. Yoshikawa, H.; Weiland, C.; Brooks-Gunn, J.; Burchinal, M.R.; Espinosa, L.M.; Gormley, W.T.; Ludwig, J.; Magnuson, K.A.;

145. Sarama, J.; Clements, D.H.; Baroody, A.J.; Kutaka, T.S.; Chernyavskiy, P.; Shi, J.; Cong, M. Testing a theoretical assumption of a learning-trajectories approach in teaching length measurement to kindergartners. *AERA Open* **2021**, *7*, 23328584211026657. [CrossRef]
146. Baroody, A.J.; Eiland, M.D.; Clements, D.H.; Sarama, J. Does a learning trajectory facilitate the learning of early cardinal-number concepts? 2021, submitted. *Eur. Polym. J.* **2021**. *submitted*.
147. Clements, D.H.; Sarama, J.; Baroody, A.J.; Joswick, C. Efficacy of a learning trajectory approach compared to a teach-to-target approach for addition and subtraction. *ZDM Math. Educ.* **2020**, *52*, 637–648. [CrossRef]
148. Clements, D.H.; Sarama, J.; Baroody, A.J.; Joswick, C.; Wolfe, C.B. Evaluating the efficacy of a learning trajectory for early shape composition. *Am. Educ. Res. J.* **2019**, *56*, 2509–2530. [CrossRef]
149. Baroody, A.J.; Yilmaz, N.; Clements, D.H.; Sarama, J. Evaluating a basic assumption of learning trajectories: The case of early patterning learning. *J. Math. Educ.* **2021**, *13*, 8–32.
150. National Research Council. *Mathematics Learning in Early Childhood: Paths toward Excellence and Equity*; Cross, C., Woods, T., Schweingruber, H., Eds.; National Academy Press: Washington, DC, USA, 2009. [CrossRef]
151.

173. Clarke, D.M.; Cheeseman, J.; Gervasoni, A.; Gronn, D.; Horne, M.; McDonough, A.; Montgomery, P.; Roche, A.; Sullivan, P.; Clarke, B.A.; et al. *Early Numeracy Research Project Final Report*

203. Martin, T.; Lukong, A.; Reaves, R. The role of manipulatives in arithmetic and geometry tasks. *J. Educ. Hum. Dev.* **2007**, *1*, 27–50. [CrossRef]
204. Griffiths, R.; Back, J.; Gifford, S. *Using Manipulatives in the Foundations of Arithmetic*; University of Leicester: Leicester, UK, 2017.
205. DeLoache, J.S.; Miller, K.F.; Rosengren, K.; Bryant, N. The credible shrinking room: Very young children's performance with symbolic and nonsymbolic relations. *Psychol. Sci.* **1997**, *8*, 308–313. [CrossRef]
206. Björklund, C. Less is more—Mathematical manipulatives in early childhood education. *Early Child Dev. Care* **2014**, *184*, 469–485. [CrossRef]
207. Brownell, W.A.; Moser, H.E. *Meaningful M8 8.G [(Claiipulatives)-261J0 3laiip0(S.cK.7A261J0S [(;).g[)]TJ0.0io [(;de0.0io III.g[)]TSubtracJ/F17 8.9 266.*

231. Hiebert, J.C. Relationships between research and the NCTM Standards. *J. Res. Math. Educ.* **1999**, *30*, 3–19. [CrossRef]
232. Kutscher, B.; Linchevski, L.; Eisenman, T. From the Lotto game to subtracting two-digit numbers in first-graders. In Proceedings of the 26th Conference of the International Group for the Psychology in Mathematics Education, Norwich, UK, 21–26 July 2002; Cockburn, A.D., Nardi, E., Eds.; University of East Anglia: Norwich, UK, 2002; Volume 3, pp. 249–256.
233. McClain, K.; Cobb, P.; Gravemeijer, K.P.E.; Estes, B. Developing mathematical reasoning within the context of measurement. In *Developing Mathematical Reasoning in Grades K-12*; Stiff, L.V., Curcio, F.R., Eds.; National Council of Teachers of Mathematics: Reston, VA, USA, 1999; pp. 93–106.
234. Karademir, A.; Akman, B. Preschool inquiry-based mathematics in practice: Perspectives of teachers and parents. *J. Qual. Res. Educ.* **2021**. [CrossRef]
235. Roche, A.; Gervasoni, A.; Kalogeropoulos, P. Factors that promote interest and engagement in learning mathematics for low-achieving primary students across three learning settings. *Math. Educ. Res. J.* **2021**, 1–32. [CrossRef]
236. Sarama, J.; Clements, D.H. Promoting a good start: Technology in early childhood mathematics. In *Learning Mathematics in the 21st Century: Adding Technology to the Equation*; Arias, E., Cristia, J., Cueto, S., Eds.; Inter-American Development Bank: Washington, DC, USA, 2020; pp. 181–223.
237. Silander, M.; Moorthy, S.; Dominguez, X.; Hupert, N.; Pasnik, S.; Llorente, C. *Using Digital Media at Home to Promote Young children's Mathematics Learning: Results of a Randomized Controlled Trial*; Society for Research on Educational Effectiveness: Evanston, IL, USA, 2016; p. 10.
238. Gunderson, E.A.; Ramirez, G.; Levine, S.C.; Beilock, S.L. New directions for research on the role of parents and teachers in the development of gender-related math attitudes: Response to commentaries. *Sex Roles* **2012**, *66*, 191–196. [CrossRef]
239. Beleslin, T.P.e.; Lepičnik-Vodopivec, J.; Partalo, S.; Sindić, A. Where does mathematics education start? Connecting the preschool curriculum and the home environment. *Our School* **2022**, *6*, 119–140. [CrossRef]
240. Sonnenschein, S.; Sun, S. Racial/ethnic differences in kindergartners' reading and math skills: Parents' knowledge of children's development and home-based activities as mediators. *Infant Child Dev.* **2017**, *26*, e2010. [CrossRef]
241. Conica, M. Domain-specific and cross-domain effects of the home literacy and numeracy environment at 3 years on children's academic competencies at 5 and 9 years. *Dev. Psychol.* **2023**, *59*, 1045–1058. [CrossRef] [PubMed]
242. Gürgah Oğul, İ.; Aktaş Arnas, Y.; Sarıbaş, Ş. Enriching mothers' maths talk with their children through home visits. *Erken Çocukluk Çalışmaları Derg.* **2020**, *4*, 833–857. [CrossRef]
243. Susperreguy, M.I.; Douglas, H.; Xu, C.; Molina-Rojas, N.; LeFevre, J.-A. Expanding the Home Numeracy Model to Chilean children: Relations among parental expectations, attitudes, activities, and children's mathematical outcomes. *Early Child. Res. Q.* **2020**, *50*, 16–28. [CrossRef]
244. Vasilyeva, M.; Laski, E.; Veraksa, A.; Weber, L.; Bukhalenkova, D. Distinct pathways from parental beliefs and practices to children's numeric skills. *J. Cogn. Dev.* **2018**, *19*, 345–366. [CrossRef]
245. Levine, S.C.; Gibson, D.J.; Berkowitz, T. Mathematical development in the early home environment. In *Cognitive Foundations for Improving Mathematical Learning*; Geary, D.C., Berch, D.B., Koepke, K.M., Eds.; Academic Press: San Diego, CA, USA, 2019; Volume 5, pp. 107–142.
246. Eason, S.H.; Nelson, A.E.; Dearing, E.; Levine, J. Facilitating young children's numeracy talk in play: The role of parent prompts. *J. Exp. Child Psychol.* **2021**, *207*, 105124. [CrossRef]
247. Linder, S.M.; Emerson, A. Increasing family mathematics play interactions through a take-home math bag intervention. *J. Res. Child. Educ.* **2019**, *33*, 323–344. [CrossRef]